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ON THE HODOGRAPH METHOD FOR AXISYMMETRIC TRANSONIC FLOWS OF GAS

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Behavior of a transonic stream of gas perturbed by a body of revolution is investigated at some distance from that body in the hodograph plane. An asymptotic expansion of the Legendre potential is derived.

The flow of a perfect gas stream, whose velocity at infinity is constant and close to the speed of sound, past a slender body of revolution is considered. The problem of attenuation of perturbations induced by the body of revolution in the transonic stream in the region upstream of compression shocks at some distance from the body is analyzed.

An asymptotic expansion of the velocity potential in the considered region was obtained in [1] in variables of the physical plane of flow. However hodograph variables proved to be more convenient in a number of problems, since the equation of shock wave in these variables becomes determinate. Because of this the asymptotics of an axisymmetric transonic stream is derived here in hodograph variables. The approximate Kármán equation for the potential of perturbed velocity is used in the analysis which is based on the method developed in [2]. An asymptotic expansion of the Legendre potential which has the required property of regularity in the hodograph plane is derived. The problem of retaining the regularity of obtained solution in its mapping onto the physical plane is investigated.

1. Let the velocity v_∞ at infinity of a stream of perfect gas flowing past a body of revolution be close to the speed of sound a_∞. We introduce a cylindrical system of coordinates x, r, directing the x-axis along the axis of symmetry. We assume the motion of gas to be everywhere isentropic and use the approximate Kármán equation [3] for the potential Φ (x, r) of perturbed velocity

$$-\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}=0 \qquad (1.1)$$

for defining the flow in the considered region. In this equation function Φ (x, r) and the variables x and r are taken in the dimensionless form. Equation (1.1) is derived on the assumption that the potential Φ (x, r) is a small addition to the potential $a_* x$ of the uniform stream whose velocity is equal to the critical speed a_*.

The principal term of the asymptotic law of attenuation of perturbations induced in a uniform sonic stream by a body of revolution is represented by the self-similar function [4 – 6]

$$\Phi_0\,(x,\ r)=r^{-2/7}f_0\,(\xi),\qquad \xi=x\,/\,r^{4/7} \qquad (1.2)$$

In the case of transonic velocity of the unperturbed stream, function Φ_0 (x, r) is also used as the principal term in the solution of the Kármán equation in the problem of flow past bodies of revolution, and the solution is sought in the form

$$\Phi\,(x,\ r)=\Phi_0\,(x,\ r)+\Phi_1\,(x,\ r) \qquad (1.3)$$

with the assumption that in the investigated region $|\,\Phi_1\,|\ll|\,\Phi_0\,|$.

We pass in Eq. (1.1) to the hodograph variables $u=\Phi_x$ and $v=\Phi_r$. To do this we introduce the Legendre potential

$$\varphi\,(u,\ v)=ux+vr-\Phi\,(x,\ r),\quad x=\varphi_u\,(u,\ v),\quad y=\varphi_v\,(u,\ v) \qquad (1.4)$$

Applying transformation (1.4) to Eq. (1.1), we obtain for function φ (u, v) the equation

$$-u\varphi_{uv}+\varphi_{uu}+v\varphi_v^{-1}\,(\varphi_{uu}\,\varphi_{vv}-\varphi_{uv}^{\ 2})=0 \qquad (1.5)$$

By analogy with the form of solution in the physical plane [1] it is reasonable to seek the solution of Eq. (1.5), which defines the transonic stream at some distance from the body, in the form

$$\varphi\,(u,v)=\varphi_0\,(u,v)+\varphi_1\,(u,v),\quad \varphi_1\,(u,v)=\sum_{k=1}^{\infty}c_{\alpha_k}\Omega_{\alpha_k}(\eta)\,v^{\alpha_k},\ \ \eta=\frac{u^3}{v^2} \qquad (1.6)$$

Using the parametric representation of function f_0 (ξ) [5 – 7]

$$f_0=8\cdot 9^{-1}s^{1/7}\,(6+3s-2s^2),\qquad \xi=s^{-2/7}\,(1-2s)$$

we pass to the selection of new hodograph variables.

In the considered approximation $s=0$ corresponds to the axis of symmetry $r=0$, and $s = {}^6/_5$ to the limit characteristics. As in [2] we introduce the new variable

$$(2^{-7/3} r s^{-1/2})^{-6/7} = \exp \sigma \tag{1.7}$$

In the considered region, i.e. for considerable r, $\exp \sigma$ is small ($\sigma \to -\infty$). Taking into consideration (1.7) for the perturbed velocity components which correspond to solution (1.2) we obtain

$$u = 2 \cdot 3^{-1} (s - 1) \exp \sigma, \quad v = 2 \cdot 9^{-1} s^{1/2} (2s - 3) \exp (3\sigma / 2) \tag{1.8}$$

We shall consider Eqs. (1.8) as transformation formulas from the hodograph variables u and v to the new independent hodograph variables s and σ. The Jacobian of this transformation shows that there is a one-to-one correspondence between variables u and v, and s and σ If variables s and σ are substituted for u and v by formulas (1.8), the variable $\eta = u^3 / v^2$ is a function of only s, and the Legendre potential expansion (1.6) assumes the form

$$\varphi (s, \sigma) = \varphi_0 (\sigma) + \varphi_1 (s, \sigma) \tag{1.9}$$

$$\varphi_0 (\sigma) = c_0 \exp (\sigma / 3), \quad \varphi_1 (s, \sigma) = \sum_{k=1}^{\infty} c_k \exp (\nu_k \sigma) \chi_k (s)$$

The first term in the right-hand part of the first formula (1.9) is the principal term which determines in the hodograph plane the law of attenuation of perturbations induced in a uniform sonic stream by a body of revolution. The second term takes into account the flow field variation induced by the deviation of the unperturbed stream velocity from the speed of sound.

It is assumed that in the considered region $| \varphi_1 (s, \sigma) | \ll | \varphi_0 (\sigma) |$. However at infinity in the physical plane ($r \to \infty$, $\sigma \to -\infty$) function φ_1 must exceed the principal term, since otherwise the continuation of solution $\varphi (s, \sigma)$ into infinity would yield a stream with sonic velocity at infinity. The last condition imposes on the selection of the range of ν_k the following restriction

$$\ldots < \nu_k < \nu_{k-1} < \ldots < \nu_1 < {}^1/_3 \tag{1.10}$$

Furthermore we shall consider only negative values of ν_k, since at a considerable distance from the body the predominant part in the first formula (1.9) is played by terms containing negative ν_k. Equation (1.5) after the introduction in it of the new independent variables s and σ assumes the form

$$\begin{aligned} & 9 \cdot 2^{-1} s (2s - 3) (\varphi_{ss} \varphi_{\sigma\sigma} - \varphi_{s\sigma}^2) - 9 \cdot 2^{-1} s (s - 1) (3s - 4) \times \\ & \varphi_s \varphi_{ss} + 9s (3s - 4) \varphi_s \varphi_{s\sigma} + 27 \cdot 2^{-1} (s - 1) \varphi_s \varphi_{\sigma\sigma} + \\ & 9 \cdot 2^{-1} s (s - 1) \varphi_\sigma \varphi_{ss} - 9s \varphi_\sigma \varphi_{s\sigma} - 27 \cdot 2^{-1} \varphi_\sigma \varphi_{\sigma\sigma} + 9 \cdot 2^{-1} (-6s^2 + \\ & 9s - 2) \varphi_s^2 + 9 \cdot 2^{-1} (3s - 1) \varphi_s \varphi_\sigma + 9 \cdot 2^{-1} \varphi_\sigma^2 \equiv R (\varphi, \varphi) = 0 \end{aligned} \tag{1.11}$$

which is accurate to within the multiplier $s^{5/2} \exp (-7\sigma / 2)$ which is neglected because in the investigated region it is nonzero.

It will be seen from (1.11) that the nonlinear operator $R (\varphi, \varphi)$ is the finite sum of quadratic operators $R^{ik} (\varphi, \varphi)$

$$R (\varphi, \varphi) = \sum_{i, k} R^{ik} (\varphi, \varphi) = \sum_{i, k} L^i (\varphi) L^k (\varphi)$$

where $L^i(\varphi)$ and $L^k(\varphi)$ are linear operators which obviously do not contain the variable σ. Substituting into the operator R function $\varphi(s, \sigma)$ from the first formula of (1.9) and allowing for the linearity of operators $L^i(\varphi)$ and $L^k(\varphi)$, we obtain

$$R(\varphi_0+\varphi_1,\ \varphi_0+\varphi_1) = R(\varphi_0,\ \varphi_0) + T(\varphi_0,\ \varphi_1) + R(\varphi_1,\ \varphi_1) \quad (1.12)$$

$$T(\varphi_0, \varphi_1) = \sum_{i,k} 2^{-1}\,[L^i(\varphi_0)\,L^k(\varphi_1) + L^i(\varphi_1)L^k(\varphi_0)]$$

where $T(\varphi_0,\ \varphi_1)$ denotes the part of operator $R(\varphi_0+\varphi_1,\ \varphi_0+\varphi_1)$, which remains after the elimination from it of operators $R(\varphi_0,\ \varphi_0)$ and $R(\varphi_1,\ \varphi_1)$. Function $\varphi_0(\sigma)$ is the solution of the transonic approximation of the equation for the Legendre potential, hence in formula (1.12) $R(\varphi_0, \varphi_0) \equiv 0$ and Eq. (1.11), after substitution into it of solution (1.9) assumes the form

$$\sum_{k=1}^{\infty} c_k T(c_0 \exp(\sigma/3),\ \chi_k(s)\exp(\nu_k\sigma)) = \quad (1.13)$$

$$- \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} c_k c_l R(\chi_k(s)\exp(\nu_k\sigma),\ \chi_l(s)\exp(\nu_l\sigma))$$

A direct substitution will prove that operators T and R in Eq. (1.13) are

$$T(c_0\exp(\sigma/3),\ \chi_k(s)\exp(\nu_k\sigma)) = \exp[({}^1/_3 + \nu_k)\sigma]\ L(\nu_k, \chi_k(s)) \quad (1.14)$$

$$R(\chi_k(s)\exp(\nu_k\sigma),\ \chi_l(s)\exp(\nu_l\sigma)) = \exp[(\nu_k+\nu_l)\sigma]\ P(\nu_k, \nu_l,\ \chi_k,\ \chi_l)$$

where L and P denote the linear and the quadratic differential operators, respectively. Substituting expressions (1.14) into Eq. (1.13) and multiplying by $\exp(-\sigma/3)$, we obtain the latter in the form

$$\sum_{k=1}^{\infty} c_k L(\nu_k, \chi_k(s))\exp(\nu_k\sigma) = -\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} c_k c_l P(\nu_k, \nu_l, \chi_k(s), \chi_l(s)) \times \quad (1.15)$$

$$\exp(\nu^{k,l}\sigma),\quad \nu^{k,l} = -\frac{1}{3} + \nu_k + \nu_l$$

Note that from all terms of this equation containing function $\chi_k(s)$ (for some fixed k), the term in the left-hand part contains $\exp\sigma$ of the lowest power. Collecting in Eq. (1.15) terms with equal coefficients $\exp \alpha_i \sigma$ and equating these to zero, we obtain the equation for determing function $\chi_i(s)$. Two alternatives may be possible for each k, viz. (1.16) or (1.17)

$$L(\nu_k,\ \chi_k(s)) = 0 \quad (1.16)$$

$$c_k\, L(\nu_k, \chi_k(s)) = -c_m c_n P(\nu_m, \nu_n, \chi_m(s),\ \chi_n(s)) \quad (1.17)$$

Equation (1.17) is valid if $\nu^{m,n} = -{}^1/_3 + \nu_m + \nu_n$ in the index of the exponent in the right-hand part of Eq. (1.15) is the same as some index in the left-hand part of the equation. Taking into consideration that all ν_k are negative, we conclude that $|\nu^{m,n}|$ is always greater than the absolute value of those ν from which $\nu^{m,n}$ are formed. The right-hand part of Eq. (1.17) contains functions χ_m and χ_n with subscripts $m, n < k$, hence (for fixed k) it is at every stage a known function of s.

We denote by ν_{0i} the values of parameters ν for which the homogeneous equation (1.16) is solvable, and by $\nu_1, \ldots, \nu_k, \ldots$ the ordered set consisting of numbers ν_{0i} and $\nu^{m,n}$ arranged according to their increasing absolute values.

Solving Eq. (1.16) with related boundary conditions represents a problem of eigenvalues. Equation (1.17) is nonhomogeneous and can be solved when $\nu^{m,n}$ are not eigenvalues of the corresponding homogeneous problem.

Let us now pass to the determination of eigenvalues of Eq. (1.16). Using the definition of operator L appearing in (1.14), we find that Eq. (1.16) has the form of the hypergeometric differential equation

$$L(\nu, \chi_\nu(s)) = 2^{-1}\{s(5s-6)\chi''(s) + 6\chi'(s)[s(2-\nu) - 1] + 3\chi(s)\nu(1-3\nu)\} = 0 \quad (1.18)$$

For solving this equation we use for boundary conditions the reasonable assumption of the absence of solution singularity at singular points $s = 0$ and $^6/_5$ to which in the physical plane correspond the axis of flow symmetry and the limit characteristic. The process of determining eigenvalues and eigenfunctions of Eq. (1.18) is described in detail in [2], where it is applied to the problem of flow of a sonic gas stream past a body of revolution. Eigenvalues of ν lying to the right of point $^1/_3$ along the number axis were also determined in that paper. Unlike in that problem, here we are interested in that part of the range of eigenvalues ν which lies to the left of point $^1/_3$ along the numer axis, i.e. the part which satisfies condition (1.10).

The solution of Eq. (1.18) is a hypergeometric function which is regulat at the point $s = {}^6/_5$ for the following values of ν:

$$\nu_{0i} = -6^{-1}[2i - 1 + (24i^2 + 24i + 1)^{1/2}], \quad i = 1, 2, \ldots \quad (1.19)$$

Formula (1.19) determines the eigenvalues of Eq.(1.18). For these values of ν the hypergeometric series which are eigenfunctions $\chi_i(s)$ degenerate into polynomials. The derived eigenvalues ν_{0i} are not sufficient for obtaining the sought expansion of the Legendre potential in powers of $\exp\sigma$, since in the series calculated by formula (1.19) values of $\nu^{m,n}$ are wedged-in between values of ν_{0i}. Functions $\chi^{m,n}(s)$ corresponding to these are to be determined by the solution of the nonhomogeneous equation (1.17) with boundary conditions for the regularity of solution at points $\sigma = 0$ and $^6/_5$.

We pass to the computation of $\nu^{m,n}$. We have

$$\nu^{1,1} = -{}^1/_3 + 2\nu_{01}$$

Owing to the validity of inequalities

$$|\nu_{01}| < |\nu_{02}| < |\nu^{1,1}| < |\nu_{03}|$$

we assign to ν_1, ν_2 and ν_3 the following values:

$$\nu_1 = \nu_{01}, \quad \nu_2 = \nu_{02}, \quad \nu_3 = \nu^{1,1} \equiv -{}^1/_3 + 2\nu_{01} \quad (1.20)$$

According to the definition of $\nu^{m,n}$

$$\nu^{1,2} = -{}^1/_3 + \nu_1 + \nu_2, \quad \nu^{1,3} = -{}^1/_3 + \nu_1 + \nu_3, \quad \nu^{2,2} = -{}^1/_3 + 2\nu_2, \quad \nu^{2,3} = -{}^1/_3 + \nu_2 + \nu_3, \quad \nu^{3,3} = -{}^1/_3 + 2\nu_3$$

Absolute values of these quantities satisfy the inequalities

$$|\nu_3| < |\nu_{03}| < |\nu^{1,2}| < |\nu^{1,3}| < |\nu_{04}| < |\nu^{2,2}|$$

Because of this we can extend sequence (1.20) as follows:

$$\nu_4 = \nu_{03}, \quad \nu_5 = \nu^{1,2} \equiv -{}^1/_3 + \nu_{01} + \nu_{02}, \quad \nu_6 = \nu^{1,3} \equiv -{}^2/_3 + 3\nu_{01}, \quad \nu_7 = \nu_{04}$$

In solving Eq. (1.17) it is reasonable to assume that

$$c_k = c_m c_n \tag{1.21}$$

Equation (1.17) now assumes the form

$$L\,(\nu^{m,n},\ \chi^{m,n}(s)) = -P\,(\chi_m(s),\ \chi_n(s)) \tag{1.22}$$

whose right-hand side is a known function in the form of a polynomial. Using this condition it is possible to show by the method of induction [2] that functions $\chi^{m,n}(s)$ which satisfy Eq. (1.22) are also polynomials.

For the derived ν_k the expansion of the Legendre potential (1.9) is of the form

$$\begin{aligned}\varphi(s,\ \sigma) = {} & c_0 \exp(\sigma/3) + c_1\chi_1(s)\exp(\nu_{01}\sigma) + c_2\chi_2(s)\exp(\nu_{02}\sigma) + \\ & c_3\chi_3(s)\exp\,[(-{}^1/_3 + 2\nu_{01})\sigma] + c_4\chi_4(s)\exp(\nu_{03}\sigma) + \\ & c_5\chi_5(s)\exp\,[(-{}^1/_3 + \nu_{01} + \nu_{02})\,\sigma] + c_6\chi_6(s)\exp\,[(-{}^2/_3 + \\ & 3\nu_{01})\,\sigma] + c_7\chi_7(s)\exp(\nu_{04}\sigma) + \ldots\end{aligned} \tag{1.23}$$

where the indices ν_{0i} are determined by formula (1.19). In this expansion all functions $\chi_i(s)$ are known polynomials. Some of the constants c_i, namely those which in expansion (1.23) appear at the eigenfunctions of the homogeneous equation (1.16), are arbitrary and have to be determined by the boundary conditions of a specific problem. Remaining constants appearing in terms originating from the nonhomogeneous equation (1.17) must be determined by formula (1.21).

Let us pass in expansion (1.23) from the variables s and σ to variables η and υ, using for this the transformation formulas (1.8). For the variable η we obtain

$$\eta = u^3/\upsilon^2 = 6s^{-1}(s-1)^3(2s-3)^{-2}$$

which shows that η depends only on the variable s. Hence $\chi_i(s)$ are functions of the variable η. Using one of the formulas (1.8), we obtain

$$\exp(\nu_k\sigma) = 9^{2/3\nu_k} \cdot 2^{-2/3\nu_k}\upsilon^{2/3\nu_k}s^{-1/3\nu_k}(2s-3)^{-2/3\nu_k} \tag{1.24}$$

Taking into account that $s = s(\eta)$ and substituting in expansion (1.23) expression (1.24) for the various powers $\exp\sigma$, we obtain

$$\begin{aligned}\varphi(s,\sigma) \equiv \varphi(\eta,\upsilon) = {} & c_0Q_0(\eta)\,\upsilon^{2/9} + c_1Q_1(\eta)\,\upsilon^{2/3\nu_{01}} + c_2Q_2(\eta)\upsilon^{2/3\nu_{02}} + \\ & c_3Q_3(\eta)\upsilon^{2/3(-1/3+2\nu_{01})} + c_4Q_4(\eta)\upsilon^{2/3\nu_{03}} + c_5Q_5(\eta)\upsilon^{2/3(-1/3+\nu_{01}+\nu_{02})} + \\ & c_6Q_6(\eta)\,\upsilon^{2/3(-2/3+3\nu_{01})} + c_7Q_7(\eta)\,\upsilon^{2/3\nu_{04}} + \ldots\end{aligned} \tag{1.25}$$

$$Q_k(\eta) = 9 \cdot 2^{-1}s^{-\nu_k/3}(2s-3)^{-2\nu_k/3}\,\chi_k(s)$$

Owing to the method of its derivation, solution (1.25) is regular along the limit characteristic, hence it can be analytically continued over that characteristic. Whether this regularity of solution is maintained in its mapping onto the physical flow plane is not clear. This aspect is investigated below.

2. Let us apply the solution derived in [1] in variables of the physical plane. We recall that we seek a solution of the Kármán equation (1.1) in the form (1.3). Substituting the sum (1.3) into Eq. (1.1) and linearizing the equation for $|\Phi_1| \ll |\Phi_0|$, we obtain for function $\Phi_1(x, r)$ a homogeneous linear equation of the second order. The solution of this equation is sought in the form of series

$$\Phi_1(x, r) = \sum_{i=1}^{\infty} c_{\omega_i} r^{\omega_i} f_{\omega_i}(\xi), \tag{2.1}$$

where c_{ω_i} are constants. The solution of the problem of eigenvalues yielded in [1] the formula for indices ω_i

$$\omega_i = 7^{-1}\ (2i - 1 + \Delta_i), \quad \Delta_i = (24i^2 + 24i + 1)^{1/2}, \quad i = 1,2\ \ldots \tag{2.2}$$

and a system of functions $f_{\omega_i}(\xi)$ which ensure the continuity of velocity and other parameters along the x-axis and the limit characteristic (such functions are called natural). It was also shown that functions $f_{\omega_i}(\xi)$ are polynomials.

Let us map the physical flow plane onto the hodograph plane. For this we expand variables x and r into series in self-similar functions in hodograph variables. In accord- ance with (1.2), (1.3) and (2.1) the perturbation velocity components are defined in the physical plane by

$$u = \frac{\partial\Phi}{\partial x} = r^{-6/7}\,[F_0(\xi) + r^{\omega_1+2/7}F_1(\xi) + r^{\omega_2+2/7}F_2(\xi) + \ldots] \tag{2.3}$$

$$v = \frac{\partial\Phi}{\partial r} = r^{-9/7}\,[G_0(\xi) + r^{\omega_1+2/7}G_1(\xi) + r^{\omega_2+2/7}G_2(\xi) + \ldots] \tag{2.4}$$

Using (2.3) and (2.4), for the self-similar variable η we obtain the expansion

$$\eta = \frac{u^3}{v^2} = \frac{F_0^3(\xi)}{G_0^2(\xi)}\,[1 + H_1(\xi)\,r^{\omega_1+2/7} + H_2(\xi)\,r^{\omega_2+2/7} + H_3(\xi)\,r^{2(\omega_1+2/7)} + \tag{2.5}$$
$$H_4(\xi)\,r^{\omega_3+2/7} + H_5(\xi)\,r^{\omega_1+\omega_2+4/7} + H_6(\xi)\,r^{3(\omega_1+2/7)} + \ldots]$$

In virtue of (2.4) the variable r can be represented in the form

$$r = a_0(\xi)\,v^{-7/9} + r_1(\xi, v), \quad r_1(\xi, v) = \sum_{k=1}^{\infty} a_k(\xi)\,v^{-\beta_k} \tag{2.6}$$

where $a_0(\xi)\,v^{-7/9}$ is the principal term and the sum $r_1(\xi, v)$ in the considerd region is small in comparison with the principal term. To determine exponents β_k we substitute expansion (2.6) into the right-hand part of formula (2.4). Using the condition of equality of the right- and left-hand parts of the derived identity, we determine successively all exponents β_k. The expansion of r in powers of v then assumes the form

$$r = a_0(\xi)\,v^{-7/9} + a_1(\xi)\,v^{-1-7/9\,\omega_1} + a_2(\xi)\,v^{-1-7/9\,\omega_2} + a_3(\xi)\,v^{-11/9-14/9\,\omega_1} + \tag{2.7}$$
$$a_4(\xi)\,v^{-1-7/9\,\omega_3} + a_5(\xi)\,v^{-11/9-7/9(\omega_1+\omega_2)} + a_6(\xi)\,v^{-13/9-21/9\,\omega_1} + \ldots$$

However this is not the final expansion, since the coefficients depend on the variable ξ and not on the hodograph variable. The expansion of ξ in hodograph variables is derived below.

The substitution of (2.7) into formula (2.5) yields

$$\eta = h_0(\xi) + h_1(\xi)\,v^{-7/9(\omega_1+2/7)} + h_2(\xi)\,v^{-7/9(\omega_2+2/7)} + h_3(\xi)\,v^{-14/9(\omega_1+2/7)} + \tag{2.8}$$
$$h_4(\xi)\,v^{-7/9(\omega_3+2/7)} + h_5(\xi)\,v^{-4/9-7/9(\omega_1+\omega_2)} + h_6(\xi)\,v^{-21/9(\omega_1+2/7)} + \ldots$$

$$h_0(\xi) = F_0^3(\xi)\, G_0^{-2}(\xi) \tag{2.9}$$

Taking into consideration expansion (2.8) we represent ξ in the form

$$\xi = \xi_0(\eta) + \xi_1(\eta, v), \quad \xi_0(\eta) = h_0^{-1}(\eta) \tag{2.10}$$

$$\xi_1(\eta, v) = \sum_{k=1}^{\infty} g_k(\eta)\, v^{\gamma_k}$$

where h_0^{-1} denotes the inverse of operator h_0. Function $\xi_0(\eta)$ is the principal term in the expansion (2.10), and function $\xi_1(\eta, v)$ is assumed to be small in comparison with $\xi_0(\eta)$. To determine exponents γ_i we substitute expression (2.10) into the right-hand part of expansion (2.8). We note that the variable ξ appears in (2.8) only as the argument of functions $h_0(\xi)$, $h_1(\xi)$, ... The values of these functions at point ξ defined by formula (2.10) can be determined by expanding them into Taylor series in the neighborhood of point ξ_0

$$h_i\left(\xi_0 + \sum_{k=1}^{\infty} g_k(\eta)\, v^{\gamma_k}\right) = h_i(\xi_0) + h_i'(\xi_0) \sum_{k=1}^{\infty} g_k(\eta)\, v^{\gamma_k} + \ldots \quad (i = 0, 1, 2) \tag{2.11}$$

with

$$h_0(\xi_0) = h_0(h_0^{-1}(\eta)) = \eta \tag{2.12}$$

Comparison of the left-and right-hand parts of expansion (2.8) with allowance for (2.11) and (2.12) successively yields exponents $\gamma_1, \gamma_2, \ldots$. The expansion of ξ in the hodograph plane now assumes the form

$$\xi = h_0^{-1}(\eta) + b_1(\eta)\, v^{-7/9(\omega_1 + 2/7)} + b_2(\eta)\, v^{-7/9(\omega_2 + 2/7)} + b_3(\eta)\, v^{-14/9(\omega_1 + 2/7)} + \tag{2.13}$$
$$b_4(\eta)\, v^{-7/9(\omega_3 + 2/7)} + b_5(\eta)\, v^{-4/9 - 7/9(\omega_1 + \omega_2)} + b_6(\eta)\, v^{-21/9(\omega_1 + 2/7)} + \ldots$$

Using a formula similar to (2.11) we expand in formula (2.7) the operators $a_0, a_1, \ldots$ for ξ defined by formula (2.13) and for the expansion of r in hodograph variables we finally obtain

$$r = R_0(\eta)\, v^{-7/9} + R_1(\eta)\, v^{-1 - 7/9\omega_1} + R_2(\eta)\, v^{-1 - 7/9\omega_2} + R_3(\eta)\, v^{-11/9 - 14/9\omega_1} + \tag{2.14}$$
$$R_4(\eta)\, v^{-1 - 7/9\omega_3} + R_5(\eta)\, v^{-11/9 - 7/9(\omega_1 + \omega_2)} + R_6(\eta)\, v^{-13/9 - 21/9\omega_1} + \ldots$$

The object of these transformations is to obtain an asymptotic expansion of the Legendre potential in the hodograph plane by using the known solution in the physical plane. As implied by formula (1.4), to do this it is necessary in addition to the expansion of r to have the expansion of the variable x and of the perturbed velocity potential Φ in the hodograph plane. We omit the cumbersome computations and present these expansions in their final form

$$x = \xi r^{4/7} = X_0(\eta)\, v^{-4/9} + X_1(\eta)\, v^{-6/9 - 7/9\omega_1} + X_2(\eta)\, v^{-6/9 - 7/9\omega_2} + \tag{2.15}$$
$$X_3(\eta)\, v^{-8/9 - 14/9\omega_1} + X_4(\eta)\, v^{-6/9 - 7/9\omega_3} + X_5(\eta)\, v^{-8/9 - 7/9(\omega_1 + \omega_2)} + X_6(\eta)\, v^{-10/9 - 21/9\omega_1} + \ldots$$

$$\Phi = r^{-2/7} f_0(\xi) + \sum_{i=1}^{\infty} c_{\omega_i} r^{\omega_i} f_{\omega_i}(\xi) = E_0(\eta)\, v^{2/9} + E_1(\eta)\, v^{-7/9\omega_1} +$$
$$E_2(\eta)\, v^{-7/9\omega_2} + E_3(\eta)\, v^{-2/9 - 14/9\omega_1} + E_4(\eta)\, v^{-7/9\omega_3} + E_5(\eta)\, v^{-2/9 - 7/9(\omega_1 + \omega_2)} +$$
$$E_6(\eta)\, v^{-4/9 - 21/9\omega_1} + \ldots$$

Formulas (2.14) and (2.15) make it possible to present the expansion of the Legendre potential in the form

$$\varphi(\eta, v) = \Omega_0(\eta)\, v^{2/9} + \Omega_1(\eta)\, v^{-7/9\omega_1} + \Omega_2(\eta)\, v^{-7/9\omega_2} + \Omega_3(\eta)\, v^{-2/9-14/9\omega_1} + \qquad (2.16)$$
$$\Omega_4(\eta)\, v^{-7/9\omega_3} + \Omega_5(\eta)\, v^{-2/9-7/9(\omega_1+\omega_2)} + \Omega_6(\eta)\, v^{-4/9-21/9\omega_1} + \Omega_7(\eta)\, v^{-7/9\omega_4} + \ldots$$

where ω_i are defined by formulas (2.2).

This expansion is the result of mapping onto the hodograph plane the solution which in the physical plane of flow is regular along the limit characteristic. Let us compare it with the similar expansion of the Legendre potential (1.25) obtained by solving the problem in hodograph variables.

It will be seen from formulas (1.19) and (2.2) that the relationship between exponents ν_{0i} and ω_i is defined by

$$\nu_{0i} = -7\omega_i/6, \quad i = 1, 2, \ldots \qquad (2.17)$$

which shows that exponents at the variable v in expansions (1.25) and (2.16) are the same. Condition (2.17) and the law of formation of exponents of series (2.16) which was obtained from the condition of solution regularity at the limit characteristic of the physical plane make it possible to conclude that solution (1.25) has also the property of regularity.

Note that the asymptotic representation of the velocity potential (1.3) derived in [1] by linearizing the Kármán equation with respect to function $\Phi_1(x, r)$ was used above. It appeared, however, that this solution mapped onto the hodograph plane is of the same form as (1.25) which was obtained by solving the nonlinearized equation which is an exact analog of the Kármán equation for the Legendre potential.

If the terms which are nonlinear with respect to function $\Phi_1(x, r)$ were taken into account in the Kármán equation (1.1), terms containing r in power $2\omega_1 + {}^2/_7$, $\omega_1 + \omega_2 + {}^2/_7$, ... (these additional exponents are formed according to the law $\omega^{m,n} = \omega_m + \omega_n + {}^2/_7$) would appear in the asymptotic formula (2.1). However this solution mapped onto the hodograph plane is also of the form (2.16), the only difference being in the coefficients $\Omega_3, \Omega_5, \Omega_6, \ldots$.

The conformity of solutions in the physical and the hodograph planes of the problem of flow of a sonic stream of gas past a body of revolution was established in [8].

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ON THE DRAG OF BODIES OF REVOLUTION AT TRANSONIC SPEEDS

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Dependence of the pattern of variation of gas parameters on the deviation of the oncoming stream velocity from sonic is established as the result of investigation of flow at great distances from bodies of revolution. This dependence makes it possible to determine the law of drag variation at transonic speeds, which is confirmed by calculations presented here.

The weak effect of the oncoming stream velocity on the deviation of parameters at the body upstream of the compression shock from their values at sonic speed at infinity is a property of transonic flows, known as the law of stabilization. It was discovered experimentally and expounded in [1] for plane flows. The relation of the stabilization law to the pattern of the stream at great distances upstream of a compression shock was established in [2, 3]. In the first of these the assumption is made that the drag is weakly dependent also on the velocity at infinity, which is not supported by experimental data. The latter reveal a rapid motion of the compression shock toward the body trailing edge, when the velocity of the oncoming stream approaches the speed of sound. For constant parameters upstream of the shock the drag is affected not only by the motion of the shock itself, but also by parameters downstream of it. For the determination of the dependence of the drag of a body on the oncoming stream velocity, it is, consequently, necessary to investigate the flow downstream of the shock.

1. Let us briefly state the properties of sonic flows at great distances from a body of revolution, which will be required subsequently. They were investigated in [4 – 11] and provide a fairly complete picture of the flow as a whole. In particular, they clarify the nature of incipient formation of drag of a body at sonic velocity.

Since investigations [4 – 11] imply that at great distances from a body the compression shock intensity is low, hence there exists a velocity potential which can be repre-